NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 5

Fall 2024

Problem 5

If $a * b = ab + a + b$ (e.g. $2 * 3 = 6 + 2 + 3 = 11$), then calculate the value of

$$Q = (1 * (2 * (3 * (4 * \cdots * (2024 * 2025) \cdots))).$$

Solution. We will show that $Q = 2026! - 1$. First, observe that the $*$ operation is associative. That is

$$a * (b * c) = a * (bc + b + c) = abc + ab + ac + a + bc + b + c = (ab + a + b) * c = (a * b) * c.$$

Hence we can rewrite Q simply as

$$1 * 2 * 3 * 4 * \cdots * 2024 * 2025.$$

Next consider the quantity

$$Q_n = 1 * 2 * 3 * 4 * \cdots * (n - 1).$$

We will show that $Q_n = n! - 1$ by induction on n, and hence $Q = Q_{2026} = 2026! - 1$. For the base case, note that $Q_2 = 1 = 2! - 1$.

For the inductive step, suppose that $Q_k = k! - 1$, and we will calculate Q_{k+1}. We have

$$Q_{k+1} = Q_k * k = (k! - 1) * k = k \cdot (k! - 1) + k! - 1 + k = (k + 1)k! - 1 = (k + 1)! - 1,$$

as desired. We conclude that $Q_n = n! - 1$, and hence $Q = 2026! - 1$.